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Commutative algebra for cohomology rings of classifying spaces of compact Lie groups

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Abstract

We apply the techniques of highly structured ring and module spectra to prove a duality theorem for the cohomology ring of the classifying space of a compact Lie group. This generalizes results of Benson–Carlson [2, 3] and Greenlees [10] in the case of finite groups. In particular, we prove a functional equation for the Poincaré series in the oriented Cohen–Macaulay case. © 1997 Elsevier Science B.V.

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1. Introduction

Some time ago,[1] based on joint work with Carlson [3] on finite group cohomology, the first author made the following conjecture. Let G be a compact Lie group, BG its classifying space, and k any field of coefficients. Then, provided that $H^*(BG;k)$ is Cohen–Macaulay, the Poincaré series

$$p_G(t) = \sum_{i\geq 0} t^i \dim_k H^i(BG;k),$$

regarded as a rational function of t, satisfies the functional equation

$$p_G(1/t) = t^d(-t)^r p_G(t).$$

Here, $d = \dim(G)$ denotes the dimension of G as a manifold, and $r = r_p(G)$ denotes the maximal rank of a finite elementary abelian p-subgroup of G if $\mathrm{char}(k) = p$ is

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[1] The Summer of 1991, to be precise.

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a prime, and the Lie rank $r_0(G)$ if $\mathrm{char}(k) = 0$; Quillen has shown this is the Krull dimension of $H^*(BG;k)$. In particular, this conjecture implies that if $H^*(BG;k)$ is Cohen–Macaulay then it is Gorenstein.

Even if $H^*(BG;k)$ is not Cohen–Macaulay, the conjecture goes on to say that for any choice of a homogeneous set of parameters $\zeta_1,\dots,\zeta_r \in H^*(BG;k)$ with ζ_i in codegree[2] n_i, there is a spectral sequence of the form described in [3], converging to the cohomology of a finite Poincaré duality complex of formal dimension $\dim(G) + \sum_{i=1}^r (n_i - 1)$. The results of [3] verify the conjectures when G is finite, but the methods do not appear to extend.

In the meanwhile, also for finite groups G, the second author [10] applied the methods of [3] to construct another spectral sequence using Grothendieck's local cohomology of $H^*(BG;k)$ with respect to the ideal J of elements of positive codegree, and converging to $H_*(BG;k)$. This gives the same information in the Cohen–Macaulay case, and is closely related to what happens in the spectral sequence of [3] in the limit as the generators are replaced by higher and higher powers.

In fact, it turns out that the conjecture is false, but for subtle reasons to do with orientation. The simplest counterexample is the orthogonal group $O(2)$ over a field k with $\mathrm{char}(k) \neq 2$. The problem comes from the fact that the adjoint representation $\mathrm{Ad}(G)$ of G is not orientable. In this paper k will denote an arbitrary commutative Noetherian ring unless otherwise stated, and we describe a spectral sequence which gives a sort of global duality for the ring $H^*(BG;k)$. In case $\mathrm{Ad}(G)$ is orientable, the statement is as follows.

Theorem 1.1. *If G is a compact Lie group of dimension d with the property that the adjoint representation $\mathrm{Ad}(G)$ is orientable over the ring k, there is a spectral sequence of the form*

$$H_J^{*,*}(H^*(BG;k)) \Longrightarrow \Sigma^{-d} H_*(BG;k).$$

Here, Σ^{-d} denotes a shift of d in degree, and $H_J^{*,*}$ denotes local cohomology with respect to J (we recall the definition in Section 2, and the grading conventions are made explicit in Corollary 5.2). More generally, without the orientability assumption, the spectral sequence converges to a twisted form of the homology of BG (see Theorem 5.1). Namely, the adjoint representation may be regarded as a group homomorphism $G \to O(d)$ to the orthogonal group of the tangent space at the identity. Compose this homomorphism with the determinant homomorphism $O(n) \to \{\pm 1\}$, to get a homomorphism $\lambda : G \to \{\pm 1\} \subseteq k^\times$ whose kernel is a subgroup H of index one or two in G. The subgroup H contains the connected component of the identity in G, so λ induces a homomorphism from $\pi_1(BG) \cong \pi_0(G)$ to $\{\pm 1\} \subseteq k^\times$, and hence a local system ε on BG. The spectral sequence then takes the form

$$H_J^{*,*}(H^*(BG;k)) \Longrightarrow \Sigma^{-d} H_*(BG;\varepsilon).$$

[2] Because we wish to view cohomology as homology with the degrees negated (and vice versa), we use the word *degree* to denote homological degree, and *codegree* to denote cohomological degree.

Notice that if k is a field and $H \neq G$ then k does not have characteristic two, and in this case, $H^*(BG;k)$ and $H^*(BG;\varepsilon)$ are the $+1$ and the -1 eigenspaces of the action of G/H on $H^*(BH;k)$ respectively.

We remark that there is still no known analogue for compact Lie groups of the resolutions constructed in [3] for finite groups. However, the above theorem gives enough information to deduce what we want about Poincaré series. Indeed since local cohomology detects depth, if $H^*(BG;k)$ is Cohen–Macaulay and $\mathrm{Ad}(G)$ is orientable over k the theorem states that $H^{r,*}(H^*(BG;k))$ is the $(d+r)$th desuspension of $H_*(BG;k)$. If k is a field this is the canonical module and so $H^*(BG;k)$ is also Gorenstein. It also has the following implication about Poincaré series.

Theorem 1.2. *Suppose that* $\mathrm{Ad}(G)$ *is orientable over a field* k, *and that* $H^*(BG;k)$ *is Cohen–Macaulay. Then the Poincaré series* $p_G(t) = \sum_{i\geq 0} \dim_k H^i(BG;k)$, *regarded as a rational function of* t, *satisfies the functional equation*

$$p_G(1/t) = t^{\dim(G)}(-t)^{r_p(G)} p_G(t).$$

We remark that the assumption of orientability of $\mathrm{Ad}(G)$ is satisfied whenever G is finite, or the component group of G is of odd order, or k has characteristic two. It is not satisfied for the orthogonal group $O(2)$ unless $\mathrm{char}(k) = 2$.

We use the method outlined in [9], which can be implemented in the category of highly structured module spectra over a highly structured ring spectrum introduced by [6]. It is proved in the companion paper by Elmendorf and May [8] that Borel cohomology is represented by a highly structured ring spectrum; using this, it is rather routine to complete the proof using Venkov's theorem [16, 17] that the cohomology of the classifying space is a Noetherian ring.

The rest of the paper is arranged as follows. We begin in Section 2 by recalling the algebra necessary to make sense of the statement of the main theorem. In Section 3 we illustrate the use of the theorem by giving a number of calculations. We then begin to introduce the method of proof by giving a quick summary of relevant facts about the Elmendorf–Kriz–Mandell–May category of highly structured modules. This prepares us for the proof itself; we recall the strategy from [9], and verify the relevant algebraic hypotheses in Section 5.

2. Local cohomology

In this section we summarize the basic definitions and properties of Grothendieck's local cohomology. The basic reference is [12], but an expository summary in a form suitable for our use is given in [11].

Suppose given a ring R, which is either ungraded and commutative, or graded and graded commutative, and which need not be Noetherian, and suppose given a finitely generated ideal $J = (\beta_1, \beta_2, \ldots, \beta_n)$. If R is graded the β_i are required to be homogeneous.

For any element we may consider the stable Koszul cochain complex

$$K^\bullet(\beta) = (R \to R[1/\beta])$$

concentrated in codegrees 0 and 1. Notice that we have a fibre sequence

$$K^\bullet(\beta) \longrightarrow R \longrightarrow R[1/\beta]$$

of cochain complexes. We may now form the tensor product

$$K^\bullet(\beta_1, \ldots, \beta_n) = K^\bullet(\beta_1) \otimes \cdots \otimes K^\bullet(\beta_n).$$

It is clear that this complex is unchanged if we replace some β by a power, and it is not hard to check that if we invert any element of the ideal J the complex becomes exact. Therefore, up to quasi-isomorphism $K^\bullet(\beta_1, \ldots, \beta_n)$ depends only on the radical of the ideal J, and we henceforth write $K^\bullet(J)$ for it. Notice that there is an augmentation $K^\bullet(J) \longrightarrow R$ obtained by tensoring the augmentations of the factors.

Following Grothendieck we define the local cohomology groups of an R-module M by

$$H_J^*(M) := H^*(K^\bullet(J) \otimes M).$$

It is easy to see that $H_J^0(M)$ is the submodule $\Gamma_J(M) := \{m \in M \mid J^N m = 0 \text{ for some } N\}$ of J-power torsion elements of M. If R is Noetherian it is not hard to prove directly that $H_J^*(R; \cdot)$ is effaceable and hence that local cohomology calculates right derived functors of $\Gamma_J(\cdot)$. It is clear that the local cohomology groups vanish above codegree n, but in the Noetherian case Grothendieck's vanishing theorem shows the powerful fact that they are zero above the Krull dimension of R. The other fact we shall use is that if $\beta \in J$ then $H_J^*(R; M)[1/\beta] = 0$; this is a restatement of the exactness of $K^\bullet(J)[1/\beta]$.

When R and M are graded the local cohomology group $H_J^s(M)$ is itself graded, and we write $H_J^{s,t}(M)$ for the codegree t part in the standard way.

3. Sample calculations

For the examples we restrict our attention to the case when k is a field. The first case to look at is where G is connected and the cohomology $H^*(G; k)$ of G as a manifold is an exterior algebra $\Lambda(\tilde{\zeta}_1, \ldots, \tilde{\zeta}_r)$ with $\deg(\tilde{\zeta}_i) = n_i - 1$, so that $\dim(G) = \sum_{i=1}^r (n_i - 1)$. In this case, by Théorème 19.1 of Borel [5], the cohomology $H^*(BG; k)$ is a polynomial ring on generators ζ_i of codegrees n_i. In particular, it is Cohen–Macaulay, and it is easy to check that the functional equation (Theorem 1.2) holds for

$$p_G(t) = \prod_{i=1}^r \frac{1}{1 - t^{n_i}}.$$

For a more non-trivial Cohen–Macaulay example, we can look at the spinor groups $G = \mathrm{Spin}(n)$ in characteristic two. Quillen [15] has calculated the cohomology in this case, and the answer is

$$H^*(B\mathrm{Spin}(n); \mathbb{F}_2) = \mathbb{F}_2[w_2, \ldots, w_n]/(\eta_2, \eta_3, \eta_5, \ldots, \eta_{2^{n-r-1}+1}) \otimes \mathbb{F}_2[\zeta_{2^{n-r}}].$$

Here, $\mathbb{F}_2[w_2, \ldots, w_n]$ is a polynomial ring in the $n-1$ Stiefel–Whitney classes for $BSO(n)$, $r = r_2(G)$ is roughly half of n but varies in a way that depends on the residue class of n modulo eight, $\eta_{2^{j-1}+1}$ $(1 \le j \le n-r)$ are elements of codegrees $2^{j-1}+1$ which form a regular sequence in $\mathbb{F}_2[w_2, \ldots, w_n]$, and $\zeta_{2^{n-r}}$ is an independent generator in codegree 2^{n-r}. Thus the Poincaré series is

$$p_G(t) = \frac{(1-t^2)(1-t^3)(1-t^5)\cdots(1-t^{2^{n-r-1}+1})}{(1-t^2)\cdots(1-t^n)(1-t^{2^{n-r}})}$$

and

$$\begin{aligned}\dim(G) = n(n-1)/2 &= 1+2+\cdots+(n-1)+(2^{n-r}-1)\\ &\quad -1-2-4-\cdots-2^{n-r-1},\end{aligned}$$

so that an easy check shows that the functional equation (Theorem 1.2) is again satisfied in this case.

In the Cohen–Macaulay case, $H_J^{s,*}(H^*(BG;k))$ is zero except when $s = r$ $(= r_p(G))$, and then

$$H_J^{r,*}(H^*(BG;k)) \cong \Sigma^{-(d+r)} H_*(G;k).$$

Recalling our convention that homology is just negatively graded cohomology (and the suspension is cohomological), this means that the E_2 page of the spectral sequence sits in the fourth quadrant, and consists of $H_*(BG;k)$ in the rth column, starting in degree $-d-r$ and working downwards.

Theorem 1.2 is now readily verified using the fact that in the Cohen–Macaulay case $H^*(BG;k)$ is finitely generated and free over the polynomial subring on generators $\zeta_1, \ldots, \zeta_r$ which generate an ideal with radical J. Thus the Poincaré series has the form $p_G(t) = q(t)r(t)$ where $q(t) = \prod_{i=1}^r 1/(1-t^{n_i})$ and $r(t)$ is a polynomial. The Poincaré series of $H^{r,*}(H^*(BG;k))$ is readily checked to be $t^{-n}r(t)q(1/t)$ where $n = n_1 + n_2 + \cdots + n_r$, and the Poincaré series of $\Sigma^{-(d+r)}H_*(BG;k)$ is $t^{-(d+r)}r(1/t)q(1/t)$. Hence $r(1/t) = t^{d+r-n}r(t)$, and as remarked above $q(1/t) = (-1)^r t^n q(t)$.

For a family of examples in which the orientation problem interferes with the functional equation, look at the orthogonal groups $G = O(2n)$, with k a field which does not have characteristic two. Let $H = SO(2n)$, the connected normal subgroup of index two in G. In this case, $H^*(BH;k)$ is a polynomial algebra on n generators $k[p_1, p_2, \ldots, p_{n-1}, e]$, with p_i in codegree $4i$ and e in codegree $2n$ (p_i are the Pontrjagin classes and e is the Euler class, which satisfies $e^2 = p_n$). The group $G/H \cong \mathbb{Z}/2$ acts on this ring by fixing the Pontrjagin classes and negating the Euler class.

Thus

$$H^*(BG;k) \cong H^*(BH;k)^{G/H} = k[p_1, \dots, p_n].$$

Although this is a Cohen–Macaulay ring, and even a Gorenstein ring, the dualizing class is in the wrong degree. The functional equation satisfied is

$$p_G(1/t) = t^{4n^2+2n-1}(-t)^n p_G(t),$$

whereas $\dim(G) = 4n^2 - 1$. The reason for this is that elements of G which are not in H act on the adjoint representation $\mathrm{Ad}(H) = \mathrm{Ad}(G)$ with a reverse in orientation. So instead of computing $H^*(BG;k)$, with $G/H = \pi_1(BG)$ acting trivially on k, we should make it act as the sign representation ε. Then

$$H^*(BG;\varepsilon) = k[p_1, \dots, p_n] \cdot e,$$

the free module of rank one over $H^*(BG;k)$ generated by the Euler class. The shift in degree of $2n$ effected by this takes care of the dualizing degree.

In general, the stable Koszul complex may be regarded as an E_1 page for the spectral sequence. It consists of $H^*(BG;k)$ in the zeroth column (in non-negative degrees), the direct sum of the rings obtained by inverting each ζ_i in turn in the first column, and so on, until the rth and last column consists of $H^*(BG;k)$ with all the ζ_i inverted.

For an example which is not Cohen–Macaulay, we examine the (simply connected) compact Lie group E_6 of dimension 78, in characteristic two. The cohomology was calculated by Kono and Mimura [13], and the answer is

$$H^*(BE_6;\mathbb{F}_2) = \mathbb{F}_2[y_4, y_6, y_7, y_{10}, y_{18}, y_{32}, y_{34}, y_{48}]/R,$$

where $\deg(y_i) = i$ and R is the ideal generated by y_7y_{10}, y_7y_{18}, y_7y_{34} and $y_{34}^2 + y_{10}^2y_{48} + y_{18}^2y_{32} +$ possibly $y_{34}y_{18}y_{10}y_6$.[3] The Poincaré series of this ring is

$$p_G(t) = \frac{1}{(1-t^4)(1-t^6)(1-t^{32})(1-t^{48})}\left\{\frac{1+t^{34}}{(1-t^{10})(1-t^{18})} + \frac{t^7}{1-t^7}\right\}.$$

A homogeneous set of parameters is given by the elements y_4, y_6, y_{32}, y_{48}, $y_7^{10} + y_{10}^7$ and y_{18}. The first five of these form a regular sequence, while the last is a zero divisor. So the depth of $H^*(BE_6;\mathbb{F}_2)$ is five.

Again the E_2 page is equal to the E_∞ page in the spectral sequence, and consists of zero except in columns five and six. The Poincaré series for column five is

$$\sum_{i\geq 0} t^{-i} \dim_{\mathbb{F}_2} H_J^{5,-i}(H^*(BE_6;\mathbb{F}_2))$$
$$= \frac{t^{-90}}{(1-t^{-4})(1-t^{-6})(1-t^{-7})(1-t^{-32})(1-t^{-48})},$$

[3] This is an ambiguity in the answer given by Kono and Mimura, which does not affect our Poincaré series calculations.

while the Poincaré series for column six is

$$\sum_{i\geq 0} t^{-i} \dim_{\mathbb{F}_2} H_J^{6,-i}(H^*(BE_6;\mathbb{F}_2))$$
$$= \frac{t^{-84}(1+t^{-34})}{(1-t^{-4})(1-t^{-6})(1-t^{-10})(1-t^{-18})(1-t^{-32})(1-t^{-48})}.$$

We conjecture that in general, in the oriented case where the depth and Krull dimension of $H^*(BG;k)$ differ by one, the appropriate functional equation is

$$p_G(1/t) - t^d(-t)^r p_G(t) = -(1+t)p'_G(t),$$

where

$$p'_G(t) = \sum_{i\geq 0} t^i \dim_k H_J^{r-1,-i}(H^*(BG;k)).$$

The latter would then satisfy the subsidiary functional equation

$$p'_G(t) = t^d(-t)^{r-1} p'_G(1/t).$$

These are the analogues of the functional equations given in Benson and Carlson [4] in the finite case.

4. Highly structured ring and module spectra

In this section we say the minimum amount possible to make sense of the structure of our proof, referring the reader to [6] and [11] for further details.

Our proof proceeds by considering the cohomology ring $H^*(BG;k)$ as the coefficients of an equivariant cohomology theory. In fact, for unbased G-spaces X, we may consider Borel cohomology

$$X \longmapsto H^*(EG\times_G X;k).$$

The coefficient ring is the value (namely $H^*(BG;k)$) this takes when X is a point, and the projection $X \longrightarrow *$ makes $H^*(EG\times_G X;k)$ into a module over this ring.

It will be convenient to work from now on with the reduced theory on based G-spaces X, for which we use the notation

$$b_G^*(X) := H^*(EG\times_G X, EG\times_G *;k) \cong \tilde{H}^*(EG_+ \wedge_G X;k).$$

In the based formulation the coefficient ring is the value of the theory on S^0:

$$b_G^* \cong \tilde{H}^*(BG_+;k) \cong H^*(BG;k).$$

For formal reasons, Borel cohomology is represented by a G-spectrum b in the sense that $b_G^*(X) = [X,b]_G^*$, where the right hand side denotes G-homotopy classes of maps of G-spectra in the sense of [14]. Indeed, if H represents ordinary cohomology with coefficients in k, we may build in non-trivial representations to form the G-spectrum i_*H and calculate using [14, II. 4.5]

$$[X, F(EG_+, i_*H)]_G^* = [EG_+ \wedge X, i_*H]_G^* = [EG_+ \wedge_G X, H]^*,$$

so that the G-spectrum $b = F(EG_+, i_*H)$ represents $b_G^*(\cdot)$. It is not hard to deduce, from the fact that $H^*(EG \times_G X; k)$ is a graded commutative ring, that the representing spectrum b is a commutative ring object in the homotopy category of G-spectra (a 'commutative ring G-spectrum').

The idea is that it would be useful if we could construct some form of derived category of modules over b. One could then work in this category to provide analogues of the constructions of Section 2, and hence exploit the formal properties of the algebra. The spectral sequence would then arise by taking the analogue of the homology of a filtered chain complex.

The first problem with this is that for an arbitrary commutative ring spectrum R there is no way to put an R-module structure on the mapping cone of an R-map between R-modules. The solution is to restrict the class of ring spectra R and endow them with extra structure. The problem arises from choices involved in the homotopies used to prove commutativity and associativity. The reason for these choices is that we have only worked in the homotopy category; the traditional solution is to continue as far as possible in the homotopy category and assume that these and all higher homotopies are unique up to homotopy. One thus reaches the definition of an E_∞ ring as a ring spectrum with extra coherence conditions on the commuting and associating homotopies. The more satisfying solution is to attribute the problem to premature passage to homotopy, and to ask that the spectrum R is actually a ring spectrum in a category of spectra before passage to homotopy; however this only makes sense if there is a smash product which is commutative, associative and unital before passage to homotopy. Such a category of spectra and such a smash product have recently (and unexpectedly) been constructed by Elmendorf–Kriz–Mandell–May [6], and they show that a spectrum which is an algebra over the sphere spectrum at the point set level is essentially the same as an E_∞ ring spectrum in the traditional sense. We shall be content to treat the Elmendorf–Kriz–Mandell–May category as a black box delivering constructions with certain properties we need. We shall refer to an algebra R over the sphere G-spectrum as a highly structured ring G-spectrum. Elmendorf and May [8] write S_G for the 0-sphere G-spectrum, and would thus refer to R as an S_G-algebra. When emphasis is necessary we refer to a module over R as a highly structured R-module.

Now suppose R is a highly structured ring G-spectrum and M is a highly structured module spectrum over it. Following Section 2, we shall explain how to define a highly structured module spectrum which is the analogue of the 'right derived J-power torsion functor' in the derived category: this suggests notations $R\Gamma_J M$ or $H_J(M)$, but we shall use the simpler notation $\Gamma_J M$ since in our context it is not ambiguous. Beginning with the principal case, for $\beta \in \pi_*^G R$ we define $\Gamma_{(\beta)}(R)$ by the fibre sequence

$$\Gamma_{(\beta)}(R) \longrightarrow R \longrightarrow R[1/\beta].$$

Here $R[1/\beta] = \underrightarrow{\text{holim}}(R \xrightarrow{\beta} R \xrightarrow{\beta} \cdots)$ is a module spectrum and the inclusion of R is a module map; thus $\Gamma_{(\beta)}(R)$ is a highly structured module. Analogous to the filtration at the chain level we have an R-module filtration of $\Gamma_{(\beta)}(R)$ by viewing it as $\Sigma^{-1}(R[1/\beta] \cup CR)$, where CR denotes the cone on R.

Next we define the J-power torsion spectrum [9, 11] for the sequence $\beta_1, \ldots, \beta_n$ by

$$\Gamma_{(\beta_1,\ldots,\beta_n)}(R) = \Gamma_{(\beta_1)}(R) \wedge_R \cdots \wedge_R \Gamma_{(\beta_n)}(R).$$

Using the same proof as in the algebraic case we conclude that $\Gamma_{(\beta_1,\ldots,\beta_n)}(R)$ depends only on the radical of $J = (\beta_1, \ldots, \beta_n)$; we therefore write $\Gamma_J(R)$ for it. It is then natural to define the J-power torsion spectrum of M by

$$\Gamma_J(M) := \Gamma_J(R) \wedge_R M.$$

To calculate the homotopy groups of $\Gamma_J(R; M)$ we use the product of the filtrations of $\Gamma_{(\beta_i)}(R)$ given above. Since the filtration models the algebra precisely, the homotopy spectral sequence of the filtered spectrum $\Gamma_J R$ gives us a useful means of calculation.

Lemma 4.1. *There is a spectral sequence*

$$E_2^{s,t} = H_J^{s,t}(R_*^G; M_*^G) \Rightarrow \pi_{-s-t}^G(\Gamma_J(M))$$

with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$.

The one other property we need is good behaviour under restriction. Indeed, if H is a subgroup of G we may view the highly structured ring G-spectrum R as a highly structured ring H-spectrum by neglect of structure, and the change of groups isomorphism $[G/H_+, R]_G^* = [S^0, R]_H^* = R_H^*$ allows us to construct the restriction homomorphism $res_H^G : R_G^* \longrightarrow R_H^*$ as the map induced by projection $G/H \longrightarrow G/G$. Given any ideal J of R_G^* we may consider the ideal $res_H^G J$ in R_H^* generated by the image of J, noting that this is also generated by the restrictions of any set of generators of J. With this notation, the behaviour under restriction is immediate by construction.

Lemma 4.2. *In the above situation there is an equivalence of highly structured module H-spectra*

$$res_H^G(\Gamma_J M) \simeq \Gamma_{res_H^G J}(res_H^G M).$$

5. Strategy

In this section we prove the main theorem modulo the fact that the representing spectrum b (which is only determined up to homotopy type) may be chosen to be a highly structured ring G-spectrum. This is proved by Elmendorf–May in the companion paper [8, 1.5]; in fact they prove the stronger result that the G-spectrum i_*H is weakly G-equivalent to a highly structured ring.

For any highly structured ring G-spectrum R we may take J to be the augmentation ideal $J = \mathit{ker}(\mathit{res}^G_1 : R^G_* \to R_*)$, and attempt to implement the strategy below; there are only three places where further assumptions are necessary, but for definiteness we shall take $R = b$ throughout, referring the reader to [11] for further discussion of the general case. By definition the coefficient ring is the ring of interest to us,

$$b^*_G = \tilde{H}^*(BG_+; k) \cong H^*(BG; k),$$

and the augmentation ideal J is the ideal of positive degree elements. Since b^*_G is Noetherian by Venkov's finite generation theorem [16, 17], the ideal J is finitely generated, and we may construct the spectrum $\Gamma_J(b)$. For the rest of the section we work entirely with highly structured modules over b.

Since $\mathit{res}^G_1(J) = (0)$ and, since for any R the augmentation gives an equivalence $\Gamma_{(0)}R \simeq R$, it follows from Lemma 4.2 that the natural augmentation $\Gamma_J(b) \longrightarrow b$ is non-equivariant equivalence, so that $EG_+ \wedge \Gamma_J(b) \simeq EG_+ \wedge b$; the map collapsing EG to a point thus gives a map

$$\kappa : EG_+ \wedge b \longrightarrow \Gamma_J b$$

whose mapping cone is $\tilde{E}G \wedge \Gamma_J(b)$, where $\tilde{E}G$ is the mapping cone of the projection $EG_+ \longrightarrow S^0$.

The main theorem is proved by showing that κ is a G-equivalence. The point is that the homotopy groups of the codomain are calculated by the spectral sequence of Lemma 4.1, whereas those of the domain are closely related to the homology of BG. In fact, since $EG_+ \wedge b = EG_+ \wedge F(EG_+, i_*H) \simeq EG_+ \wedge i_*H$, it is immediate from the Adams isomorphism [14, II. 7.2] that

$$\pi^G_*(EG_+ \wedge b) = \tilde{H}_*(EG_+ \wedge_G S^{\mathrm{Ad}(G)}; k),$$

where $\mathrm{Ad}(G)$ is the adjoint representation, and $S^{\mathrm{Ad}(G)}$ is its one point compactification with the new point as its G-fixed basepoint.

Theorem 5.1. *For any compact Lie group G and commutative Noetherian ring k there is a spectral sequence*

$$E_2^{s,t} = H_J^{s,t}(H^*(BG;k)) \Rightarrow \tilde{H}_{-s-t}(EG_+ \wedge_G S^{\mathrm{Ad}(G)}; k)$$

of modules over $H^*(BG;k)$ with differentials $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$.

We say that the adjoint representation is *orientable* over k if it can be replaced by the G-fixed representation of the same dimension so that we have an isomorphism $\tilde{H}_*(EG_+ \wedge_G S^{\mathrm{Ad}(G)}; k) \cong \tilde{H}_*(\Sigma^d BG_+; k)$ of modules over $H^*(BG;k)$. A Serre spectral sequence argument shows this is the case if G acts trivially on $H^d(S^{\mathrm{Ad}(G)}; k)$, which is certain if G is finite, the component group is of odd order or k is of characteristic two. However, the adjoint representation may not be orientable; for example that of $O(2)$ is not orientable over k unless $\mathrm{char}(k) = 2$.

Corollary 5.2. *For any compact Lie group G of dimension d and commutative Noetherian ring k over which the adjoint representation is orientable, there is a spectral sequence*

$$E_2^{s,t} = H_J^{s,t}(H^*(BG;k)) \Rightarrow H_{-s-t-d}(BG;k)$$

of modules over $H^*(BG;k)$ with differentials $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$.

It remains to prove that κ is a G-equivalence, or equivalently that its mapping cone is G-contractible. Since all descending chains of subgroups in a compact Lie group are finite, we may suppose by induction that the analogous statement has been proved for all proper subgroups H of G. To make use of this assumption we need to know augmentation ideals are compatible in a sense we now make precise.

Lemma 5.3. *For any subgroup H of G the augmentation ideals $J(G)$ of $H^*(BG;k)$ and $J(H)$ of $H^*(BH;k)$ are related by*

$$\sqrt{res_H^G(J(G))} = J(H).$$

Proof. The proof of Venkov's finite generation theorem [16, 17] shows that the restriction map $H^*(BG;k) \longrightarrow H^*(BH;k)$ is finite, and the assertion follows. Indeed if $\mathfrak{p}$ is a prime of $H^*(BH;k)$ containing $res_H^G J(G)$ then $(res_H^G)^{-1}(\mathfrak{p}) \supseteq J(G) = (res_H^G)^{-1}(J(H))$; by the Going Up Theorem $\mathfrak{p} \supseteq J(H)$. $\square$

It follows from Lemma 4.2 that we have equivalences of H-spectra

$$res_H^G \Gamma_{J(G)} b \simeq \Gamma_{res_H^G J(G)} b \simeq \Gamma_{J(H)} b,$$

so that we may safely write J without qualification. In particular, by untwisting [14, II.4.8] and the inductive hypothesis we have

$$G/H_+ \wedge \tilde{E}G \wedge \Gamma_J b \simeq G \ltimes_H (\tilde{E}H \wedge \Gamma_J b) \simeq *$$

for any proper subgroup H of G, and hence

$$T \wedge \tilde{E}G \wedge \Gamma_J b \simeq *$$

whenever T is built out of cells $G/H_+ \wedge S^n$ with H a proper subgroup.

The extreme example of such a space T is the space $E\mathscr{P}_+$. Here $E\mathscr{P}$ is the universal space for the family $\mathscr{P}$ of proper subgroups characterized by the property that $E\mathscr{P}$ is H-contractible for any proper subgroup H, but $(E\mathscr{P})^G = \emptyset$. The cofibre sequence

$$E\mathscr{P}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathscr{P}$$

and the inductive hypothesis show that it is enough to prove that $\tilde{E}\mathscr{P} \wedge \Gamma_J b$ is G-contractible. Since $\tilde{E}\mathscr{P}$ is H-contractible for any proper subgroup H, it follows from the Whitehead theorem that it is enough to show $\pi_*^G(\tilde{E}\mathscr{P} \wedge \Gamma_J b) = 0$.

At this point we must recall that for any complex representation V there is a Thom class $t(V) \in \tilde{H}^{|V|}(EG_+ \wedge_G S^V; k)$, giving rise to Thom isomorphisms

$$\tilde{H}^{n-|V|}(EG_+ \wedge_G X; k) \xrightarrow{\cong} \tilde{H}^n(EG_+ \wedge_G \Sigma^V X; k)$$

by external multiplication. In particular, taking $X = S^0$, the pullback of the unit is the Euler class $\chi(V) \in b_G^{|V|} = \tilde{H}^{|V|}(BG_+; k)$; equivalently the inclusion $e(V) : S^0 \to S^V$ gives a diagram

$$\begin{array}{ccc} b_G^n(S^V \wedge X) & \xrightarrow{e(V)^*} & b_G^n(S^0 \wedge X) \\ \cong \big\uparrow \text{Thom} & & \big\| \\ b_G^{n-|V|}(X) & \xrightarrow{\cdot\chi(V)} & b_G^n(X). \end{array}$$

The represented manifestation of the Thom isomorphism is a G-equivalence $S^V \wedge b \simeq S^{|V|} \wedge b$.

Using this, there is a useful reduction.

Lemma 5.4 (Carlsson's reduction). *It is sufficient to show* $\pi_*^G(S^{\infty V} \wedge \Gamma_J(b)) = 0$ *for a single chosen complex representation* V *provided* $V^G = 0$.

Proof. Since $V^G = 0$, we have an equivalence $S^{\infty V} \wedge \tilde{E}\mathscr{P} \simeq \tilde{E}\mathscr{P}$, so that it is enough to show $\pi_*^G(S^{\infty V} \wedge \tilde{E}\mathscr{P} \wedge \Gamma_J(b)) = 0$. However $\tilde{E}\mathscr{P}$ can be constructed as a direct limit of spheres S^W where W runs over complex representations without trivial summand. Since $S^W \wedge \Gamma_J(b) \simeq S^{|W|} \wedge \Gamma_J(b)$ from the Thom isomorphism, the hypothesis ensures that $\pi_*^G(S^{\infty V} \wedge S^W \wedge \Gamma_J(b)) = 0$, and hence the direct limit of these groups is also zero. $\square$

It is now easy to complete the proof of Theorem 5.1; indeed we may calculate

$$\begin{aligned} \pi_*^G(S^{\infty V} \wedge \Gamma_J(b)) &= \varinjlim_k \pi_*^G(S^{kV} \wedge \Gamma_J b) \\ &= \varinjlim \{\pi_*^G \Gamma_J b, \chi(V)\} \\ &= (\pi_*^G \Gamma_J b)[1/\chi(V)]. \end{aligned}$$

But $\chi(V) \in J$ since $e(V)$ is non-equivariantly null-homotopic and so

$$H_J^*(b_*^G)[1/\chi(V)] = 0;$$

from the spectral sequence of Lemma 4.1 we see that

$$\pi_*^G(S^{\infty V} \wedge \Gamma_J(b)) = 0. \qquad \square$$

References

[1] J.F. Adams, Prerequisites (on equivariant stable homotopy) for Carlsson's lecture, Lecture Notes in Mathematics, Vol. 1051 (Springer, Berlin, 1984).

[2] D.J. Benson and J.F. Carlson, Complexity and multiple complexes, Math. Z. 195 (1987) 221–238.

[3] D.J. Benson and J.F. Carlson, Projective resolutions and Poincaré duality complexes, Trans. Amer. Math. Soc. 342 (1994) 447–488.

[4] D.J. Benson and J.F. Carlson, Functional equations for Poincaré series in group cohomology, Bull. London Math. Soc., to appear.

[5] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. Math. 57 (1953) 115–207.

[6] A. Elmendorf, I. Kriz, M. Mandell and J.P. May, Rings, Modules and Algebras in Stable Homotopy Theory, Amer. Math. Soc. Surveys and Monographs, to appear.

[7] A. Elmendorf, I. Kriz, M. Mandell and J.P. May, Modern foundations for stable homotopy theory, in: I.M. James, Ed., Handbook of Topology (North-Holland, Amsterdam, 1995).

[8] A. Elmendorf and J.P. May, Algebras over equivariant sphere spectra, J. Pure Appl. Algebra 116 (1997) 139–149.

[9] J.P.C. Greenlees, The K-homology of universal spaces and local cohomology of the representation ring, Topology 32 (1993) 295–308.

[10] J.P.C. Greenlees, Commutative algebra in group cohomology, J. Pure Appl. Algebra 98 (1995) 151–162.

[11] J.P.C. Greenlees and J.P. May, Completions in algebra and topology, in: I.M. James, Ed., Handbook of Topology (North-Holland, Amsterdam, 1995).

[12] A. Grothendieck (notes by R. Hartshorne), Local Cohomology, Lecture Notes in Mathematics, Vol. 41 (Springer, Berlin, 1967).

[13] A. Kono and M. Mimura, Cohomology mod 2 of the classifying space of the compact connected Lie group of type E_6, J. Pure Appl. Algebra 6 (1975) 61–81.

[14] L.G. Lewis, J.P. May and M. Steinberger (with contributions by J.E. McClure), Equivariant Stable Homotopy Theory, Lecture Notes in Mathematics, Vol. 1213 (Springer, Berlin, 1986).

[15] D.G. Quillen, The mod 2 cohomology rings of extra-special 2-groups and the spinor groups, Math. Ann. 194 (1971) 197–212.

[16] B.B. Venkov, Cohomology algebras for some classifying spaces, Dokl. Akad. Nauk. SSSR 127 (1959) 943–944.

[17] B.B. Venkov, Characteristic classes for finite groups, Dokl. Akad. Nauk. SSSR 137 (1961) 1274–1277.